Research Lesson Proposal for Grade 8/9 Algebra: Functions

For the lesson on February 3, 2012
At Prieto Math and Science Academy, Chicago
Instructor: Tom McDougal
Proposal developed by: Andrew Friesema and Tom McDougal

1. Title of the Lesson: Inventing Function Notation

2. Summary of the lesson

Using the context of prices at two different stores, the lesson will motivate the need to give functions names and to create an efficient notation for functions. Students will be challenged to devise their own notation and then to compare theirs with other notation proposals. They will then learn the standard mapping and $f(x)$ notations.

3. Goals of the Lesson

a. For students to understand why it is useful to name functions.
b. For students to appreciate the merits of function notation as a concise way to express the relationship between a function, its input, and the corresponding output.
c. For students to grow in their ability to analyze, critique, and discuss mathematical ideas with each other.

4. Relationship of the Unit to the Standards

<u>Prior related standards:</u>
7.RP 2c Represent proportional relationships by equations.
7.RP 3 Use proportional relationships to solve multistep ration and percent problems.
7.EE 3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers…
7.EE 4 Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
F.BF 3 Identify the effect on the graph of replacing f(x) by f(x)+k… (for linear functions only, using equations in x and y).

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This unit

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Standards addressed in later units or grades:
F.IF 4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of a relationship.
F.IF 8b Use properties of exponents to interpret expressions for exponential functions

F.IF 8a Use… factoring and completing the square… to find zeros, extreme value, and symmetry of the graph [of a quadratic function] and interpret these in terms of a context.
F.IF 8b Use properties of exponents to interpret expressions for exponential functions.

5. Background and Rationale

These 8th grade students are voluntarily taking Algebra as an extra morning class. They were hand-picked on the basis of their 7th grade ISAT scores (usually 80th percentile and above), their grades, and their track record of arriving on time. Successful completion of the course and performance on the district's 8th-grade algebra exit exam allows them to place out of algebra in 9th grade at most high schools. They are also taking the regular 8th-grade math course which uses *Connected Mathematics*.

Some of the students have been informally exposed to the idea of functions since 3rd grade. In Everyday Math they saw input-output diagrams with rules stated in words. Students are also accustomed to (x, y) tables from their work in 7th grade.

In the CCSS, the concept of functions is first introduced in 8th grade as a relation in which a unique output is determined by an input. The concept also often includes an idea of dependency, where one variable can be considered dependent and the other independent, such as temperature changing over time. This is an important idea in science, and it is mentioned in the CCSS in the overview (p. 67). But the CCSS do not otherwise explicitly mention it as a learning goal for students. Nor does the CME text address it. We do not discuss dependency in this unit.

The traditional model high school pathway in Appendix A of the CCSS emphasizes linear and exponential functions in Unit 2, which is the guide for our unit. Being based on the CME text, however, our unit does not address exponential functions; those come in a later chapter.

In addition to the concept of a function, this unit will address the following standards listed for Unit 2 in the traditional model high school pathway in Appendix A of the CCSS (p. 17)

- Use $f(x)$ notation, where f denotes the function, "$f(x)$ denotes the output of the function, and the graph of f is the graph of the equation $y=f(x)$." Students should be able to evaluate functions using function notation. (F.IF 1, 2)
- Interpret functions that arise in applications in terms of a context.
- Analyze functions using different representations.
- Build a function that models a relationship between two quantities, and compose functions in a simple context (F.BF 1— n.b. composition is labeled with (+) to flag it as "additional mathematics.")
- Understand the concepts of domain and range, and the relationship of domain to the graph "and, where applicable, to the quantitative relationship it describes." (F.IF 5)
- Calculate and interpret the rate of change of a function over a specified interval (F.IF 6)

Students taking this course in the past have generally accepted function notation and not had many difficulties with it, except with composition. Many other students do have trouble with the notation, however, especially Euler ($f(x)$) notation. Students confuse this with multiplication. Also, the equals sign in a sentence like $f(x) = 3x$ plays a different role than in equations students have seen before. The text uses mapping (arrow) notation as well as $f(x)$ notation. It does not say why. Although the Common Core does not mention this notation, we choose to include it because we think it may help remind students what a function is.

6. Research and *Kyozaikenkyu* (“investigation of materials for teaching”)

We examined the following texts for their treatments of functions:

- CME *Algebra* (the text used for this course)
- *Interactive Mathematics Program*, Year 1 (Key Curriculum Press, 2009)
- UCSMP *Advanced Algebra* (1993)
- Saxon *Algebra 2* (1983)
- A Japanese 7-9 math series from Tokyo Shoseki (in preparation)
- Advanced Algebra: A Precalculus Course (Brown and Robbins 1984)
- Mathematics 1: Japanese Grade 10 (1996, from the AMA)

Of these texts, only the Saxon and UCSMP texts provided any motivation for function notation. In both of them, function notation is presented as a way to distinguish outputs from different equations. Only the UCSMP text derives this motivation from a context.

The Japanese texts were of particular interest to us because the Japanese curriculum has been generally praised for its focus and coherence, and was one of the models for the Common Core State Standards. The Tokyo Shoseki series introduces the concept of a function in grade 7 prior to developing the ideas of direct proportion, inverse proportion, and graphs of each of these. Linear functions that are not direct proportions are developed in grade 8. The term “function” continues to be used in grades 8 and 9, but function notation and composition are not introduced until grade 10. The series also chooses to use an informal expression translated as “interval of values” to examine the ideas of domain and range, in advance of a formal treatment of those ideas in grade 10.

The sequence and pacing in Japanese curriculum make us simultaneously wonder why the word “function” is introduced so late. We also question the merit of introducing function notation, composition, and domain and range this early—especially so promptly after students learn what a function is. No other ideas covered later in Algebra require them. Where function notation is used in the CME text, it is used gratuitously. But the algebra exit exam, the model pathway for the Common Core for the traditional sequence, and most high schools expect that students will learn these ideas in this course.

The Common Core emphasizes the distinction between a function f and its output $f(x)$, but it is common practice to choose function names evocative of their output. For example, one text[1] uses $A(r) = \pi r^2$ to express the idea that the area of a circle is a function of its radius. In this example, what does A represent? Common though this practice may be, we believe it is not helpful to students who are first learning about functions and function notation, and for this introduction to function notation we have chosen a context in which the function is clearly distinct from its output.

7. About the Lesson

One of our goals in this lesson study is to better engage students in whole-class discussions. Our experience, both in our own teaching and in lessons we have observed, is that when students come to the front of the room to present, other students sit back passively. Any discussion that occurs tends to be between the presenting student(s) and the teacher.

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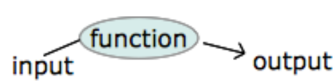
1 Benson, J. et al. (1991). *Algebra 2 and Trigonometry*. McDougal, Littell, Evanston, Ill.

These students in particular are not very forthcoming in whole-class discussions, although they do talk in their groups. The topic of this research lesson presents an extra challenge because function notation is a convention—just the way it's done—so what room for discussion does that allow?

To encourage discussion in general, we have begun using three strategies. First, where it makes sense, we will have multiple solutions or ideas—ideally, conflicting answers or ideas—clearly posted on the board *before* we ask for discussion. We will then ask students to compare the posted solutions following a think-pair-share structure, i.e. discussing with a partner or group before the discussion is opened to the whole class. Second, we will routinely ask students to restate the comments of their peers. This will serve two purposes: it hold students accountable for listening to each other, and it will also give students more than one opportunity to hear and understand the ideas, giving them more time to consider whether they agree or disagree. And third, we will often write student observations on the board, with the student's name, to communicate the importance of the contribution. We expect to use all three of these strategies in the research lesson.

For this lesson, students will be asked to invent their own function notation, an authentic mathematical task. The discussion will then focus on evaluating the different proposals (with the Euler and mapping notations mixed in) based on clarity and brevity. We will then tell them which are the standard notations, and ask them to practice using them.

8. Flow of the Unit

Lesson	Learning objective(s)
1	Students discuss a situation in which a store charges different customers a different price for the same item. In contrast to this, they learn that a *function* connects an input (e.g. an item) to exactly one output (its price). Students practice this concept by distinguishing functional and non-functional relationships from contexts. Students see a diagram like this: 
2	Students learn that many functions in mathematics can be expressed with a rule. They are challenged to come up with a rule for a function for a multi-step situation, and learn the value of working first with concrete numbers. (F.BF 1a)
3	Students play "guess the rule" for in-out tables and learn strategies for doing so, such as "divide the output by the input" (to test for a proportion).
4	**(Research lesson)** Students learn that functions can be named (with words or letters), they learn how to use the name in mapping (arrow) and Euler ($f(x)$) notation. (F.IF 1)
5	Students express composition of functions using both mapping and f(x) notations, using the example of Fred's Grocery buying their stock from Big Box. Students consider constraints on allowable inputs to a function, either from context or because certain inputs cause the function to be undefined. They determine the corresponding outputs of a function, and they learn the terms *domain* and *range*. (F.IF 5)

6	Learn to recognize, from their graphs, functions vs. non-functional relations, and identify the domain and range. (F.IF 5)
7	Recognize linear vs non-linear functions from a table of values.
8	Students learn about functions that can be defined recursively, and their application to arithmetic sequences. (F.IF 3)
9	Write recursive rules for functions that describe arithmetic sequences. (F.BF 2)

9. Flow of the Lesson

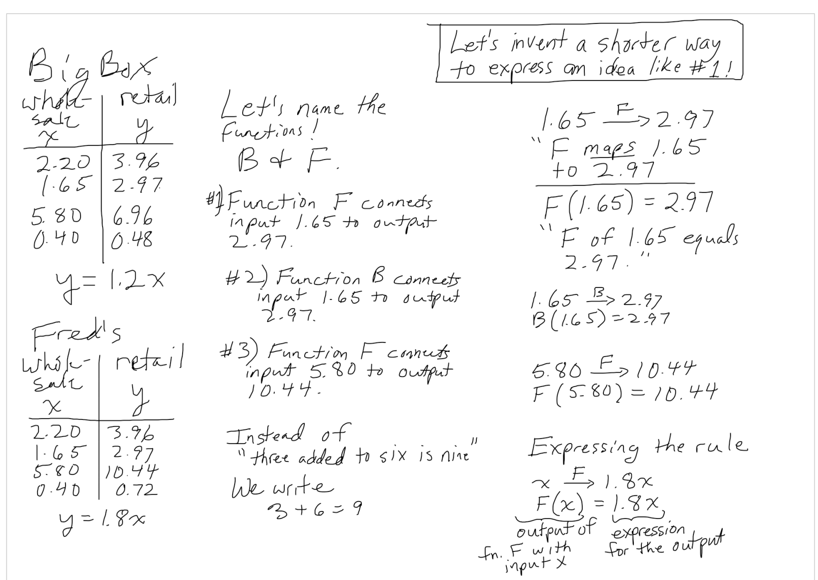
Steps, Learning Activities **Teacher's Questions and Expected Student Reactions**	**Teacher's Support**	**Points of Evaluation**
1. Introduction Revisit the examples of Big Box Market and Frank's Grocery. T: Are these functions? Why? What's the input? What's the output? S: Yes, yes, because for any input, you know exactly what the output will be. Input: wholesale price. Output: retail price. T: If the input is $4.20, what is the output? S: Which store? T: "Which store?" = "Which function?" Suppose we give them names? Function B (Big Box) and function F (Fred's). Now, if the input is $4.20, using function B, what is the output?	Post the in-out charts for the two stores, with the rules beneath.	Can Ss express why these are functions? Does this example motivate giving functions names?
Write for several examples: "Function __ connect input __ to output ___" T: "Is anyone else's hand getting tired?	"Please write these with me in your notes."	Does this motivate students to find a shorthand?
2. Posing the Task Compare to notations for numbers, addition, subtraction etc. "People invented these symbols." The task: Let's invent a shorter way to clearly express an idea like 'Function F connects input 2.20 to the output 3.96'.	Post the task. Ask Ss to think quietly for 2 minutes, jot down as many ideas as they can.	
3. Anticipated Student Responses Variations on arrow diagrams. [Fn F, I: 2.20, O: 3.96]	If a student is stuck: identify the three things to be represented (fn, input, output). Suggest a simple picture or using abbreviations.	Do students' representations contain all three elements?
4. Comparing and Discussing "Now we are going to compare our notations and choose the best ones. Let's focus on two criteria: clarity and brevity." Each group choose one & write on 8.5"x11" paper. Post all notations on the board. Mix in the conventional arrow and Euler notations.	On the board: Criteria (reasons) for choosing: - clarity (easy to see the function, the input, the output) - brevity (not too much writing) Please write BIG!	Do students use these criteria?

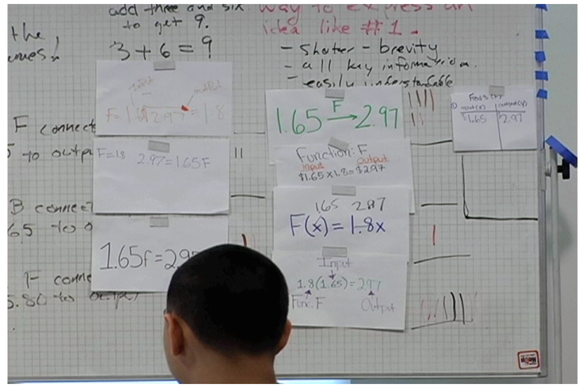
Groups discuss what is on the board. Students then vote w/ tally marks for the one they think is best. After each group has picked one, ask for justifications for 2-3 of them. "If we are going to communicate with the rest of the world, we will need to use a notation that other people understand. People have agreed on two notations." Remove all but the two conventional ones. Explain how to say them. Students repeat aloud.	Which group picked this one? Tell us why." On board 2.20 —F→ 3.96 "F maps 2.20 to 3.96" *F*(2.20) = 3.96 "F of 2.20 equals 3.96"	**Do students just choose their own?** Is the lesson helping students defend their choices?
5. Practice and Extension Ask students to rewrite each of the sentences from the introduction using this notation. Ask a student to read each of these. Show that the notation can also be used to show the rule.	x —F→ 1.8x F(x) = 1.8x	Can students use these notations
5. Summing up Main points from today: – Functions can have names, like F, B. – New ideas require new notations, so we don't have to write so much. – Functions can be expressed using mapping or F(x) notation.		

10. Evaluation

- Did the context of the two stores help students appreciate the value of naming functions?
- Did students appreciate the value of a shorthand for expressing the relationship between a function, an input, and the corresponding output?
- Did the opportunity to try to invent their own notation for that relationship help them understand and appreciate the standard $f(x)$ and mapping notations?
- Did the lesson provoke students to think critically about their own and their classmates' ideas, and did it help them express their reasoning?

11. Board Plan





12. Post-lesson reflection

During the post-lesson discussion, the following major topics were discussed:

- the impact of the particular choice of functions for Big Box and Fred's grocery
- the degree to which the lesson was effective at motivating function notation
- whether it was appropriate, effective, and worth the time to ask students to invent their own function notation
- whether the lesson made adequate space for student thinking.

<u>Impact of the choice of functions</u>

The rules for Big Box and Fred's Grocery, which students discovered in the previous lesson as part of a "guess the rule" game, were y=1.1x and y=1.8x, respectively. In this lesson, students were asked to generate their own notations to express the idea, "function F [Fred's Grocery] connects input 1.65 with output 2.97." Then, students voted on the various proposed notations based on the criteria of brevity and clarity. Standard mapping notation and f(x) notation, prepared in advance, were mixed in with the student notations.

When students voted, two notations were clear favorites: mapping notation and "1.65F = 2.97". It emerged in the lesson that, in the latter expression, students were thinking of "F" as representing the constant 1.8 from the rule y=1.8x.

Clearly, it was an egregious mistake to use two functions both of which were simple proportions, especially given that one of our goals with this lesson was to overcome the common misconception that f(x) notation denotes multiplication of two values denoted by f and x. At least one of the functions should not have been a proportion, and perhaps we should also have used a function whose rule was unknown. But it may nonetheless be useful to have at least one function that *is* a simple proportion, to permit the misconception to be brought to light and addressed.

<u>Appropriateness of asking students to invent a notation</u>

Heather Brown quoted a paragraph from *Making Sense* (Carpenter et al., 1997, pp. 36-37) which said that "students cannot be expected to discover [social conventions such as notation]." She argued that the core task, therefore, was not mathematical and not appropriate.

It was not our goal, however, that students should discover conventional notation. Rather, our primary goal was to prepare students to understand the standard notation by having students think deeply *first* about the meaning the notation is supposed to convey. Asking them to invent a notation was a way to achieve that. We also wished to humanize and demystify mathematics a little by making the point that notations are invented by people, and by helping students understand the criteria by which society eventually settles on some notations and discards others. For these reasons we believe the task was both mathematical and appropriate.

Was it worth a whole class period? In our write-up we said that students in this course in the past have not had difficulty with function notation. So were we addressing a non-existent problem? These students are the strongest math students at Prieto; even if function notation has not historically been a problem for them, several participants said that they have seen students struggle with function notation in later courses, and the prevailing opinion seemed to be that a little extra time spent helping students understand the notation well in Algebra 1 would be time well spent if it eliminated later difficulties.

Motivating function notation

We wanted students to see two benefits from having a special notation for functions: 1) it permits us to identify which function should be used when more than one is present; 2) it allows us to express the connection between a function, an input, and a corresponding output more succinctly than using words. The lesson was not obviously successful in either of these. Tom asked the students if they could say what the output was for an input of 4.20, without specifying which function they should use, yet students seemed perfectly willing to give an answer. Later, after writing three versions of "Function __ connects input __ with output __," Tom asked the students if their hand was getting tired, and most students said "No."

Having more than just two functions present, and allowing students to give (different) answers and then argue about which is correct, would help with #1. For #2, perhaps students needed to write the words more often (e.g. over the course of several days). On the other hand, perhaps it is sufficient to consider wordiness, and make the comparison to "three added to six is nine" and “3+6=9.”

Ultimately, about half the students voted for the standard arrow (mapping) notation (those who chose a different one were mostly doing so because it matched the multiplication in the rule). We are pleased with this preference, because we think the mapping notation clearly represents the function concept. But this experience made it clear to us that $f(\mathrm{x})$ notation needs to be motivated in a different way, probably through composition, which may need another lesson.

Making space for or building off of student thinking

When students began devising their own notations, most students included the rule. Tom interrupted them and told them not to. We agree with observers, including Tad Watanabe, who suggested it that it would have been better to go with what the students wrote and compare them to the original English sentences. There could then have been an opportunity to revise their ideas. It would help to include a function for which the rule was unknown.

The lesson did not successfully elicit much student discussion over the merits of the different notations, partly because there was not much motivation for students to do so. We simply asked them to explain why they chose one. More motivating would be to ask them to convince the rest of the class that their notation was the best and we should all adopt it.

We would also like to try giving students the opportunity to use the notation of their choice multiple times, perhaps over more than one day, before asking them to make a final decision.

Final thoughts

We find two reasons why students might benefit from spreading this topic over several days: students could use English for awhile (“function __ connects…”), which would help them appreciate a more succinct representation. Then, students could experiment with different notations, giving them time to weigh the benefits of one vs. another. That could extend into a lesson on composition, which would provide the motivation to switch away from arrow notation.

All this could be done while addressing other topics. Students could build functions for different situations and express those in their notation of choice.

Apart from the topic itself, we note an irony in our own lesson study practice. In his talk, Tom pointed out the importance of getting feedback from a "knowledgeable other" during the planning process. But we didn't do this! If we had, perhaps we would have been alerted to the problem of using functions that were simple multiplicative relations.

We wish to express our thanks to Bernadette Lasich, the classroom teacher who let us take over her class for a week, Tad Watanabe, who provided final comments, Heather Brown, who moderated the discussion, and to the panelists and observers who have us their time to help us critically examine and reflect upon the lesson.